Useful Formulas for Communication Systems

Revision 1.0.3

Author: Jan Molina

Text: *Modern Digital and Analog Communication Systems, 4th Ed. (B.P. Lathi, Zhi Ding)*

Chapter 2

Signal energy

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt$$

Signal power

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt$$

Signal decomposition

$$e(t) = g(t) - cx(t)$$

$$\mathrm{c} = \frac{\int_{t_1}^{t_2} g(t)x(t)dt}{\int_{t_1}^{t_2} x^2(t)dt} = \frac{1}{E_x}\int_{t_1}^{t_2} g(t)x(t)dt$$

$$g(t) \approx c \cdot x(t)$$

Signal correlation

$$\rho = \frac{1}{\sqrt{E_g E_x}} \int_{-\infty}^{\infty} g(t)x(t)dt$$

Signal correlation (General/Complex)

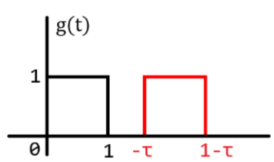
$$\rho = \frac{1}{\sqrt{E_g E_x}} \int_{-\infty}^{\infty} g(t)x^*(t)dt$$

Cross correlation

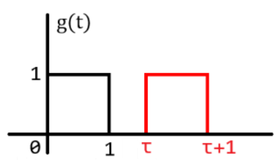
$$\psi_{zg}(\tau) = \int_{-\infty}^{\infty} z(t)g^*(t-\tau)dt = \int_{-\infty}^{\infty} z(t+\tau)g^*(t)dt$$

Auto correlation

$$\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t+\tau)dt$$



$$\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t-\tau)\,dt$$

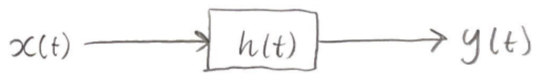


Chapter 3 (1/2)

Parseval's Theorem The energy E_g of a signal $g(t)$ can be obtained from the time domain *OR* the frequency domain. $E_g = \int_{-\infty}^{\infty} \lvert G(f)\rvert^2 \, df$
ESD (Energy Spectral density) $\Psi_g(f) = \lvert G(f)\rvert^2$ $E_g = \int_{-\infty}^{\infty} \Psi_g(f) df$ $\psi_g(\tau) \leftrightarrow \Psi_g(f)$
PSD Power spectral density $S_g(f) = \lim_{T\to\infty} \frac{\lvert G_T(f)\rvert^2}{T} = \lim_{T\to\infty} \frac{\Psi_{gT}(f)}{T}$ $P_g = \int_{-\infty}^{\infty} S_g(f) df = 2\int_0^{\infty} S_g(f) \, df$
Time autocorrelation $R(\tau)$ $R_g(\tau) = \lim_{T\to\infty} \int_{-T/2}^{T/2} g(t)g(t-\tau)dt = \lim_{T\to\infty} \frac{\psi_{gT}(\tau)}{T}$ $R_g(\tau) = R_g(-\tau)$ $R_g(\tau) \leftrightarrow S_g(f)$
Power is the "mean square" value $P_g = \widetilde{g^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t) = \lim_{T\to\infty} \frac{E_{gT}}{T}$

Chapter 3 (2/2)

Input and output PSD and ESD



$$Y(f) = H(f) \cdot X(f)$$

$$|Y(f)|^2 = |H(f)|^2 |X(f)|^2$$

Output ESD

$\Psi_y(f) = |H(f)|^2 \cdot \Psi_x(f)$

Output PSD

$S_y(f) = |H(f)|^2 \cdot S_x(f)$
$P_y = \int_{-\infty}^{\infty} S_y(f) df = 2\int_0^{\infty} S_y(f)\, df$

PSD of modulated signals

For a power signal $g(t)$ modulated by a cosine wave with frequency f_0 where $f_0 \geq B$, that is

$$\phi(t) = g(t)\cos(2\pi f_0 t),$$

the PSD and power are

$$S_\phi(f) = \frac{1}{4}\left[S_g(f + f_0) + S_g(f - f_0)\right]$$

$$P_\phi = \frac{1}{2} P_g$$

Pre-computed PSDs

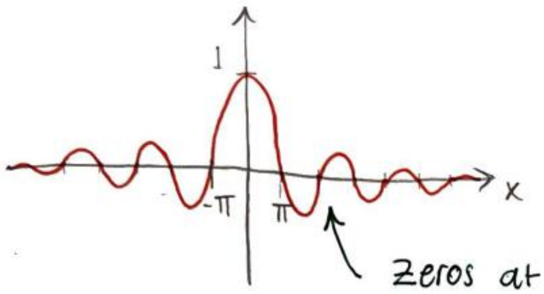
$g(t) = C \cdot \cos(2\pi f_0 t + \theta_0)$

$R_g(\tau) = \frac{C^2}{2}\cos(2\pi f_0 \tau)$

$S_g(f) = \frac{C^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$

Useful functions – Sinc

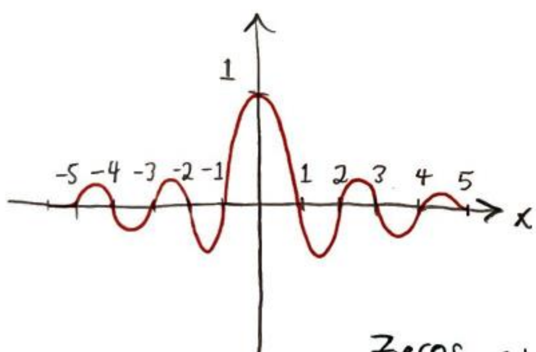
Regular sinc



$$\text{sinc}(x) = \frac{\sin(x)}{x}$$

Zeros at $n\pi$ for $n = \pm 1, \pm 2, \pm 3, \ldots$

Normalized sinc



$$\text{sinc}_{\pi}(x) = \frac{\sin(\pi x)}{\pi x}$$

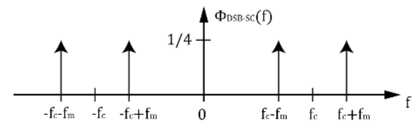
Zeros at $n = \pm 1, \pm 2, \pm 3, \ldots$

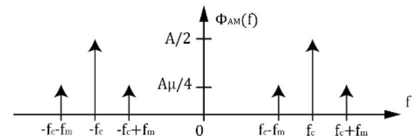
Chapter 4

DSB-SC modulated signal

$\phi_{\text{DSB-SC}}(t) = m(t)\cos(2\pi f_c t)$

$\Phi_{\text{DSB-SC}}(f) = \frac{1}{2}[M(f + f_c) + M(f - f_c)]$





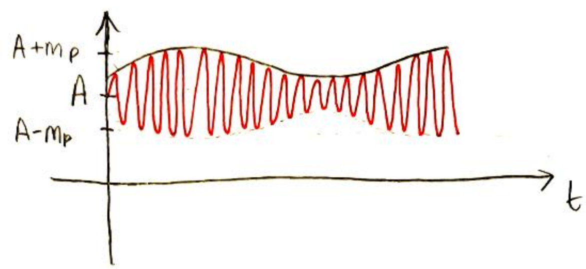
AM modulated signal

$\phi_{\text{AM}}(t) = [A + m(t)]\cos(2\pi f_c t)$

$\Phi_{\text{AM}}(f) = \frac{1}{2}[M(f + f_c) + M(f - f_c)] + \frac{A}{2}[\delta(f + f_c) + \delta(f - f_c)]$

Modulation index (AM)

$\mu = \frac{m_p}{A}$



Carrier and Sideband power

$P_c = \frac{A^2}{2}$

$P_s = \frac{1}{2}\widetilde{m^2(t)} = \frac{1}{2} \cdot \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} m^2(t)dt$

$P_{total} = P_c + P_s$

Power efficiency

$\eta = \frac{\text{useful power}}{\text{total power}} = \frac{P_s}{P_c + P_s}$

Time domain representation of SSB signals

$\phi_{USB}(t) = m(t)\cos(\omega_c t) - m_h(t)\sin(\omega_c t)$

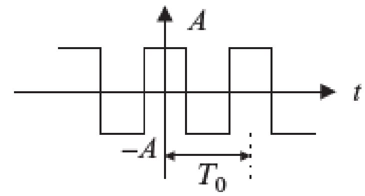
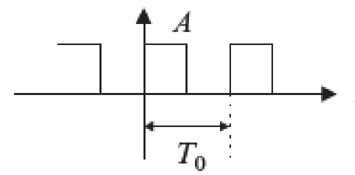
$\phi_{LSB}(t) = m(t)\cos(\omega_c t) + m_h(t)\sin(\omega_c t)$

$\phi_{SSB}(t) = m(t)\cos(\omega_c t) \mp m_n(t)\sin(\omega_c t)$

Demodulation

$\phi_{SSB}(t) \cdot 2\cos(\omega_c t) = m(t) + [m(t)\cos(2\omega_c t) \mp m_h(t)\sin(2\omega_c t)]$

Fourier Series expansions

Square wave $x(t) = \frac{4A}{\pi}\left(\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t) - \frac{1}{7}\cos(7\omega_0 t) + \cdots\right)$	
Positive square wave $x(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\sin\ \omega_0 t + \frac{1}{3}\sin\ 3\omega_0 t + \frac{1}{5}\sin\ 5\omega_0 t + \frac{1}{7}\sin\ 7\omega_0 t + \cdots\right)$	

Chapter 5 (1/2)

General Form of an Angle-modulated signal

$\phi_{EM}(t) = A\cos[\theta(t)]$

Key Attributes

A = amplitude of the angle-modulated signal

ω_c = carrier frequency $\left(f_c = \frac{\omega_c}{2\pi}\right)$

$\theta(t)$ = time-varying angle

B = bandwidth of modulating signal m(t)

m_p = amplitude or "peak value" of m(t)

B_{EM} = bandwidth of the angle-modulated signal

k_p = phase sensitivity

k_f = frequency sensitivity

Δω = frequency deviation $\left(\Delta f = \frac{\Delta\omega}{2\pi}\right)$

$\Delta\phi$ = phase deviation

Other attributes

$\omega_i(t) = \frac{d}{dt}\theta(t)$ Instantaneous frequency $\left(f_i = \frac{\omega_i}{2\pi}\right)$

$P_{avg} = \frac{A^2}{2}$ Power of an angle-modulated signal

FM-modulated signal

$\phi_{FM}(t) = A\cos\left[2\pi f_c t + k_f \int_{-\infty}^{t} m(\alpha)d\alpha\right]$

$f_i(t) = f_c + \frac{k_f}{2\pi} m(t)$

$B_{FM} = 2(\Delta f + B)$

$\Delta f = k_f \frac{m_p}{2\pi}$

PM-modulated signal

$\phi_{PM}(t) = A\cos\left[2\pi f_c t + k_p m(t)\right]$

$f_i(t) = f_c + \frac{k_p}{2\pi}\dot{m}(t)$

$B_{PM} = 2(\Delta f + B)$

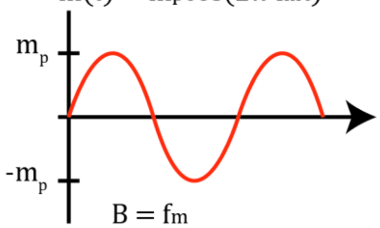
$\Delta f = k_p \frac{\left[\dot{m}(t)_{max} - \dot{m}(t)_{min}\right]}{2 \cdot 2\pi}$

Chapter 5 (2/2)

Bandwidth of m(t) signal

Case 1: Sinusoid

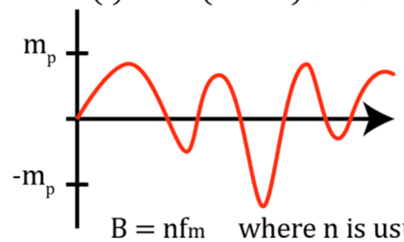
$m(t) = m_p \cos(2\pi \cdot f_m t)$



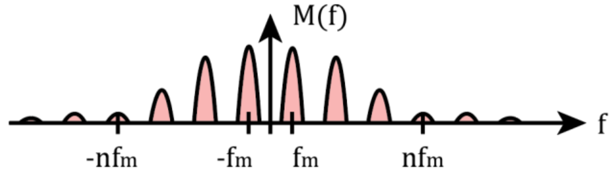
$B = f_m$

Case 2: Generic analog signal

$m(t) = \cos(2\pi \cdot f_m t) + \ldots + \cos(2\pi \cdot n f_m t) + \ldots$



$B = n f_m$ where n is usually given



Chapter 6

Nyquist Rate and Interval $R_N = 2B$ $T_N = \frac{1}{R} = \frac{1}{2B}$